Indian Statistical Institute, Bangalore

M. Math.

First Year, Second Semester

Functional Analysis

Final Examination (Back-paper) — Date: 04 June 2024

Maximum marks: 100 — Time: 10:00 AM–1:00 PM (3 hours)

Instructor: Chaitanya G K

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1. Let X be a normed linear space.
   (a) Find all subspaces of X which are contained in some ball $B_r(x)$ of X.
   (b) Find all subspaces of X which contains some ball $B_s(y)$ of X.

   [6+6]

2. Prove that a normed linear space X is Banach if and only if every absolutely convergent series in X is convergent. [13]

3. State and prove Riesz lemma. [10]

4. State and prove Open Mapping Theorem. [15]

5. Let $X = C^1[0,1]$ be the space of continuously differentiable functions on $[0,1]$ and $Y = C[0,1]$. The norm on X and Y is the sup norm. Consider the map $T : X \to Y$ defined by

$$Tf(t) = f'(t), \quad t \in [0,1].$$

   Show that the graph of T is closed but T is not bounded. Does this contradict the closed graph theorem? [13]

6. Show that every separable infinite-dimensional Hilbert space is isomorphic to l^2. [15]

7. Let (c_n) be a sequence of complex numbers. Define an operator D on l^2 by

$$Dx = (c_1x_1, c_2x_2, \ldots), \quad \forall x = (x_1, x_2, \ldots) \in l^2.$$

   Prove that D is compact if and only if $\lim_{n\to\infty} c_n = 0$. [10]

8. Let X be a Banach space, and $S, T \in \mathcal{B}(X)$. Show that if T is compact, the spectrum of S and $S+T$ are same except for eigenvalues. [12]

9. Let $\tau_{\|\cdot\|}$ and τ_w denote the norm and the weak topologies on a normed linear space X, respectively. If X is infinite dimensional, show that $\tau_w \neq \tau_{\|\cdot\|}$. [15]

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